

# Simplicity transformations for three-way arrays with symmetric slices

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# Outline

- 1 Introduction
  - Definitions. Concepts. Notation
  - Why?
  - Results available so far
- 2 Symmetric slice arrays
  - Symmetric slice  $l \times l \times K_{max}$  arrays
  - Symmetric slice  $2 \times 2 \times K$  arrays
  - Symmetric slice  $3 \times 3 \times K$  arrays
  - Symmetric slice  $4 \times 4 \times K$  arrays
- 3 Maximal simplicity
- 4 Example of application: typical rank
- 5 Conclusions. Considerations. Developments

# Definitions. Concepts. Notation

- $\underline{\mathbf{X}}$ :  $I \times J \times K$  three-way array  
 $\mathbf{X}_1, \dots, \mathbf{X}_K$ :  $I \times J$  frontal slices

$$\underline{\mathbf{X}} = [\mathbf{X}_1 | \dots | \mathbf{X}_K]$$

- 3PCA: find  $\mathbf{A}_{I \times P}$ ,  $\mathbf{B}_{J \times Q}$ ,  $\mathbf{C}_{K \times R}$ ,  $\underline{\mathbf{G}}_{P \times Q \times R}$  to minimize  $\sum_k \text{tr}(\mathbf{E}'_k \mathbf{E}_k)$

$$\underline{\mathbf{X}} = \sum_p \sum_q \sum_r g_{pqr} (\mathbf{a}_p \circ \mathbf{b}_q \circ \mathbf{c}_r) + \underline{\mathbf{E}}$$

- CP: constrained 3PCA ( $P = Q = R$ ,  $\underline{\mathbf{G}}$  superunit diagonal)

$$\underline{\mathbf{X}} = \sum_r (\mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r) + \underline{\mathbf{E}}$$

- Typical rank: minimal number of components  $R$  that allow a perfect CP decomposition

# Definitions. Concepts. Notation

- Weight of an array = # nonzero entries
- 3PCA is not unique: for any nonsingular  $\mathbf{S}_{P \times P}$ ,  $\mathbf{T}_{Q \times Q}$ ,  $\mathbf{U}_{R \times R}$

$$\begin{aligned} \mathbf{A} &\longrightarrow \mathbf{A}(\mathbf{S}')^{-1} \\ \mathbf{B} &\longrightarrow \mathbf{B}(\mathbf{T}')^{-1} \\ \mathbf{C} &\longrightarrow \mathbf{C}(\mathbf{U}')^{-1} \\ \mathbf{G}_a = [\mathbf{G}_1 | \cdots | \mathbf{G}_R] &\longrightarrow \mathbf{S}'\mathbf{G}_a(\mathbf{U} \otimes \mathbf{T}) \end{aligned}$$

Tucker transformation

# Why?

Challenge: given  $\mathbf{X}$ , find suitable  $\mathbf{S}$ ,  $\mathbf{T}$ ,  $\mathbf{U}$  such that  $\mathbf{SX}(\mathbf{U} \otimes \mathbf{T})$  has many zero entries (small weight)

Why?

- 1 Facilitate interpretation of 3PCA decompositions

Example: rotate  $\mathbf{G}$  so that several entries become zero



less interactions of components to account for during interpretation of 3PCA

- 2 Constrained 3PCA: distinguish between tautologies and non-trivial models
- 3 Mathematical applications: typical rank, maximal rank

## Results available so far (I)

- Cohen (1974, 1975), MacCallum (1976), Kroonenberg (1983): “diagonalize” frontal slices of  $\underline{\mathbf{G}}$  ( $P = Q$ )
- Kiers (1992): “super-diagonalize”  $\underline{\mathbf{G}}$  ( $P = Q = R$ )
- Kiers (1998): SIMPLIMAX  
 $\underline{\mathbf{G}} \rightarrow$  minimize ssq ( $m$  smallest elements)
- Murakami et al. (1998)  
 $P = QR - 1$   
Example:  $P = 5, Q = 3, R = 2$

$$\mathbf{G}_a = [\mathbf{G}_1 | \mathbf{G}_2] \rightarrow \left[ \begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \mu_1 & 0 & 0 & 0 & \mu_2 & 0 \end{array} \right]$$

## Results available so far (II)

- Ten Berge & Kiers (1999)

$$P \times Q \times 2, P > Q$$

$$\mathbf{G}_a = [\mathbf{G}_1 | \mathbf{G}_2] \longrightarrow \left[ \begin{array}{c|c} \mathbf{I}_Q & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_Q \end{array} \right]$$

- Ten Berge et al. (2000): Multiple orthonormality
- Rocci & Ten Berge (2002)

- ▶  $P \times P \times 2$

Example:  $P = 3$

$$\mathbf{G}_a \longrightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 0 & b \end{array} \right] \text{ or } \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & a \\ 0 & 0 & 1 & 0 & -a & 0 \end{array} \right]$$

- ▶ Orthogonal Complement Algorithm

What about symmetric slice arrays?

# Symmetric slice arrays

- $\underline{\mathbf{X}} = [\mathbf{X}_1 | \cdots | \mathbf{X}_K]$ : order  $I \times I \times K$ 
  - ▶ assume:  $\underline{\mathbf{X}}$  is randomly sampled from a continuous distribution with symmetry constraint ( $\mathbf{X}_k$  symmetric,  $\forall k$ )
  - ▶ slices  $\mathbf{X}_k$  linearly independent
  - ▶ number of slices:  $K = 1, 2, \dots, \underbrace{\frac{I(I+1)}{2}}_{K_{\max}}$

- symmetry-preserving transformation of  $\underline{\mathbf{X}}$ 
  - ▶  $\mathbf{S}_{I \times I}$ ,  $\mathbf{U}_{K \times K}$  nonsingular

$$\mathbf{H}_I = \mathbf{S}' \left( \sum_k u_{kl} \mathbf{X}_k \right) \mathbf{S}, \quad I = 1, 2, \dots, K$$

- ▶ GOAL: introduce as many zeros in  $\mathbf{H}$  as possible
- Orthogonal Complement Method: “symmetric” version

## Symmetric slice $I \times I \times K_{max}$ arrays

- {frontal slices} = basis for the space of symmetric  $I \times I$  matrices
- simple basis for the same space (Rocci & Ten Berge(1994)):  
(notation:  $\mathbf{e}_i$  = column  $i$  of  $\mathbf{I}_I$ )

$$\mathbf{e}_i \mathbf{e}_i', \quad i = 1, \dots, I$$
$$\mathbf{e}_i \mathbf{e}_j' + \mathbf{e}_j \mathbf{e}_i', \quad 1 \leq i < j \leq I$$

Example:  $I = 3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- frontal slice mix suffices

## Symmetric slice $2 \times 2 \times K$ arrays

- $K_{\max} = 3$ , so  $K = 1, 2, 3$
- $2 \times 2 \times 3$ : done ( $K_{\max}$  situation)
- $2 \times 2 \times 1$ : use EVD

$$\underline{\mathbf{X}} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}; \text{ if } \alpha < 0: \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- $2 \times 2 \times 2 =$  orthogonal complement of  $2 \times 2 \times 1$

$$\underline{\mathbf{X}} \longrightarrow \left[ \begin{array}{cc|cc} \alpha & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array} \right]; \text{ if } \alpha < 0: \left[ \begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

### Conclusion for $2 \times 2 \times 2$ :

- ▶ weight 4 is always possible
- ▶ if  $\underline{\mathbf{X}}^c$  has eigenvalues of both signs then weight 2 is possible

## Symmetric slice $3 \times 3 \times K$ arrays

- $K_{\max} = 6$ , so  $K = 1, 2, 3, 4, 5, 6$
- $3 \times 3 \times 6$ : done ( $K_{\max}$  situation)
- $3 \times 3 \times 1$ : use EVD

$$\underline{\mathbf{X}} \longrightarrow \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}; \text{ if } d_2 d_3 < 0: \begin{bmatrix} d_1 & 0 & 0 \\ 0 & 0 & 2d_2 \\ 0 & 2d_2 & 0 \end{bmatrix}$$

- $3 \times 3 \times 5 =$  orthogonal complement of  $3 \times 3 \times 1$

$$\underline{\mathbf{X}} \longrightarrow \left[ \begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$
$$\left[ \begin{array}{ccc|ccc|ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & \alpha & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

Conclusion for  $3 \times 3 \times 5$ :

- ▶ weight 10 is always possible
- ▶ if  $\underline{\mathbf{X}}^c$  has eigenvalues of both signs then weight 9 is possible

# Symmetric slice $3 \times 3 \times K$ arrays

- $3 \times 3 \times 2$ : see  $\text{EVD}(\mathbf{X}_1^{-1} \mathbf{X}_2)$

- ▶ real eigenvalues

$$\underline{\mathbf{X}} \longrightarrow \left[ \begin{array}{ccc|ccc} 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]; \text{ also: } \left[ \begin{array}{ccc|ccc} -\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array} \right]$$

- ▶ complex eigenvalues

$$\underline{\mathbf{X}} \longrightarrow \left[ \begin{array}{ccc|ccc} -\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array} \right]$$

## Conclusion for $3 \times 3 \times 2$ :

- ▶ weight 5 is always possible
- ▶ if  $\mathbf{X}_1^{-1} \mathbf{X}_2$  has real eigenvalues then weight 4 is possible

## Symmetric slice $3 \times 3 \times K$ arrays

- $3 \times 3 \times 4 =$  orthogonal complement of  $3 \times 3 \times 2$

$$\left[ \begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta\alpha & 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right]$$

▶  $3 \times 3 \times 4$

( $\delta = 1/ - 1$  in real/complex case)

### Conclusion for $3 \times 3 \times 4$ :

- ▶ weight 8 is always possible
- $3 \times 3 \times 3$ : still open!
  - ▶ when a  $3 \times 3 \times 3$  array has an orthogonal complement, it is also  $3 \times 3 \times 3 \dots$
  - ▶ simulation: a weight 9 pattern seems to be possible almost 90% of the times
  - ▶ to be continued (...)

# Symmetric slice $4 \times 4 \times K$ arrays

- $K_{\max} = 10$ , so  $K = 1, 2, \dots, 8, 9, 10$
- $4 \times 4 \times 10$ : done ( $K_{\max}$  situation)
- $4 \times 4 \times 1$ : use EVD

$$\begin{aligned} \text{in general: } \mathbf{X} &\longrightarrow \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix} \\ \text{if } d_1, d_2, d_3 > 0 \\ d_4 < 0 &\longrightarrow \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & 0 & 2d_3 \\ 0 & 0 & 2d_3 & 0 \end{bmatrix} \\ \text{if } d_1, d_3 > 0 \\ d_2, d_4 < 0 &\longrightarrow \begin{bmatrix} 0 & 2d_1 & 0 & 0 \\ 2d_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2d_3 \\ 0 & 0 & 2d_3 & 0 \end{bmatrix} \end{aligned}$$

- $4 \times 4 \times 9 =$  orthogonal complement of  $4 \times 4 \times 1$ 
  - ▶ weight 18 is always possible
  - ▶ depending on the signs of  $\text{eigs}(\mathbf{X}^c)$  we can have weight 17 or 16

## Symmetric slice $4 \times 4 \times K$ arrays

- $4 \times 4 \times 2$ : see  $\text{EVD}(\mathbf{X}_1^{-1} \mathbf{X}_2)$ 
  - ▶ real eigenvalues: weight 6

$$\left[ \begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & \gamma & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

- ▶ one pair of complex eigenvalues: weight 7

$$\left[ \begin{array}{cccc|cccc} \alpha & 0 & 0 & 0 & \gamma & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \end{array} \right]$$

- ▶ two pairs of complex eigenvalues: weight 8

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & -1 & 0 & 0 & \gamma & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \end{array} \right]$$

# Symmetric slice $4 \times 4 \times K$ arrays

- $4 \times 4 \times 8 =$  orthogonal complement of  $4 \times 4 \times 2$ 
  - ▶ any symmetric slice  $4 \times 4 \times 8$  array can almost surely be simplified into one out of two weight 18 arrays

Example: one of the targets

$$\left[ \begin{array}{cccc|cccc|cccc|cccc} \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 & 0 \\ 0 & \star & 0 & 0 & 0 & \star & 0 & 0 & 0 & \star & 0 & 0 & 0 & \star & 0 & 0 & 0 \\ 0 & 0 & 0 & \star & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[ \begin{array}{cccc|cccc|cccc|cccc} 0 & 0 & \star & 0 & 0 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & \star \\ \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 \end{array} \right]$$

# Maximal simplicity

Question: can simpler targets be found for the cases previously presented?

Answer:

- $3 \times 3 \times K$  for  $K = 1, 2, 4, 5, 6$ : NO (proved)
- $4 \times 4 \times K$  for  $K = 8, 9$ : NO(?) (simulation)

## Example of application: typical rank

- $\underline{\mathbf{X}}$ : symmetric slice  $3 \times 3 \times 4$  array
- Ten Berge et al. (2004)

$$\text{typical rank } (\underline{\mathbf{X}}) = \{4, 5\}$$

- rank=4?, rank=5?

Check if roots of a certain fourth degree polynomial are real and distinct.

Using  $\rightarrow 3 \times 3 \times 4$  simple form, and applying the same reasoning as in Ten Berge et al. (2004), we conclude that:

- rank  $(\underline{\mathbf{X}}) = 4$  iff  $\delta = 1$  and  $\alpha > 0$  (and rank is 5 otherwise)
- a CP decomposition is now straightforward

Example: rank=4

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & \sqrt{\alpha} & -\sqrt{\alpha} \\ \sqrt{\alpha} & -\sqrt{\alpha} & 0 & 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5\sqrt{\alpha^{-1}} & -0.5\sqrt{\alpha^{-1}} \\ 0.5\sqrt{\alpha^{-1}} & -0.5\sqrt{\alpha^{-1}} & 0 & 0 \end{bmatrix}$$

# Conclusions. Considerations. Developments

## Conclusions

- simplification achieved for some types of arrays with symmetric frontal slices; closed form rotation matrices available
- maximal simplicity achieved (mathematically proved or empirically verified via SIMPLIMAX)
- typical rank considerations come as nice follow-ups

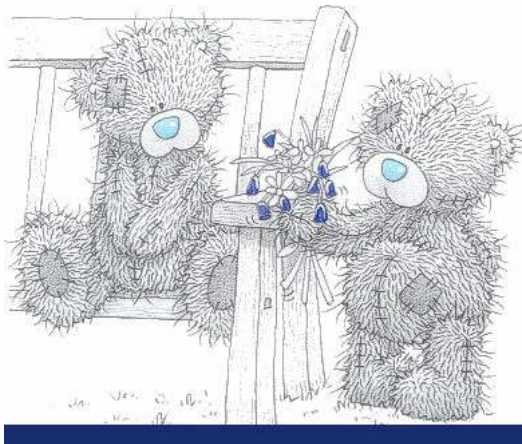
## Considerations

- 3PCA core arrays are not “randomly sampled from a continuous distribution”, but do behave as if they were
- valid contribution for Matrix Theory: simultaneous reduction of more than a pair of matrices to sparse forms is scarce

## Developments

- extend results to other orders
- address issues like: maximal simplicity, typical rank

Muchas gracias!



QUESTIONS?